Hypersurface in (pseudo) riemannian manifold

$\Sigma \xrightarrow{\iota} \iota\Sigma \subset M \qquad m = n+1$

normal and projectors

$n \quad \nu \qquad n\cdot\nu = 1 \qquad {}^{\perp}\delta = n\nu \qquad {}^{\|}\delta = \delta - n\nu$

metric, orthogonal splitting, normalization

$g = s\,\nu\nu + q \qquad n\cdot q = 0 \qquad \nu = s\, g\cdot n \qquad s = \pm 1$

${}^{\|}g = q \qquad {}^{\perp}g = s\,\nu\nu$

extrinsic curvature

$K_a{}^b = \nabla_{\|a} n^b \qquad k_{ab} = \nabla_{\|a}\nu_b$

we have

$K_{ab} = s\,k_{ab} \qquad K_{ab} = K_{ba} \qquad K_{a\perp} = 0$

related quantities

$\kappa = K_a{}^a \qquad K^2_{ab} = K_{am}K_{bn}q^{mn} \qquad \mathcal{K}^2 = K_{ab}K^{ab} = K^{2}{}_a{}^a$

second fundamental form

$\mathrm{II}^m_{ab} = -K_{ab}n^m \qquad \mathrm{II}_{am}{}^b = -K_a{}^b\nu_m$

$\mathrm{Tr}\,\mathrm{II}_m = -\kappa\,\nu_m \qquad (\mathrm{Tr}\,\mathrm{II})^2 = s\,\kappa^2 \qquad \mathrm{II}^2_{ab} = s\,K^2_{ab} \qquad \mathrm{Tr}\,\mathrm{II}^2 = s\,\mathcal{K}^2$

splitting of curvature - see general theory

$R_{\|a\|b\|c\|d} = {}^{\|}R_{abcd} - s\,(K_{ac}K_{bd} - K_{ad}K_{bc})$

$R_{\|a\|b\|c\perp} = {}^{\|}\nabla_a K_{bc} - {}^{\|}\nabla_b K_{ac}$

$Ric_{\|a\|b} = s\,R_{\perp\|a\perp\|b} + {}^{\|}Ric_{ab} - s\,(\kappa K_{ab} - K^2_{ab})$

$Ric_{\perp\|a} = {}^{\|}\nabla_c K_a{}^c - {}^{\|}\nabla_a \kappa$

$Ric_{\perp\perp} = R_{\perp\|a\perp\|b}\,q^{ab}$

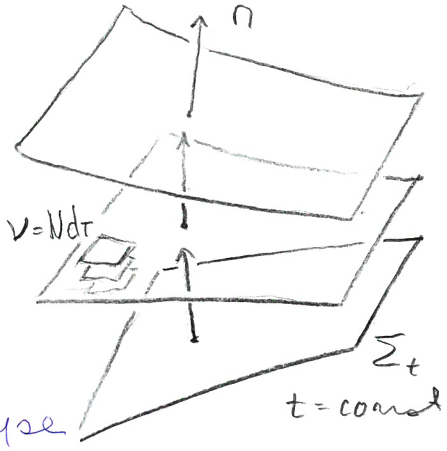
$R = 2s\,Ric_{\perp\perp} + {}^{\|}R - s\,(\kappa^2 - \mathcal{K}^2)$

$Ein_{\perp\perp} = -\frac{s}{2}\,{}^{\|}R + \frac{1}{2}(\kappa^2 - \mathcal{K}^2)$

$Ein_{\perp\|a} = {}^{\|}\nabla_c K_a{}^c - {}^{\|}\nabla_a \kappa$

what about components $R_{\perp\|a\perp\|b}$ $Ric_{\perp\perp}$?

Foliation of manifold by hypersurfaces



time function

$t: M \to \mathbb{R}$

hypersurface $\Sigma_t \Leftrightarrow t = \text{const}$

dt has a fixed causal character

spacelike $s=+1$ or timelike $s=-1$

normalized normal and lapse

$\nu_a = N d_a t \qquad \nu_a \nu_b g^{ab} = s \qquad N$ lapse

$n^a = s g^{ab} \nu_b \qquad n^a n^b g_{ab} = s \qquad n^a \nu_a = 1$

acceleration

$$a^m = n^k \nabla_k n^m \qquad a^m \nu_m = 0 \quad \Leftarrow \quad \nu^2 = s$$

$$a_m = -s \frac{1}{N} {}^{\shortparallel}d_m N \qquad {}^{\shortparallel}d_m N = -s N a_m \qquad s a_m = -{}^{\shortparallel}d_m \log N = n^k \nabla_k \nu_m$$

proof:

$$n^k \nabla_k \nu_m = n^k \nabla_k (N d_m t) = (n^k \nabla_k N) d_a t + N n^k \nabla_k \nabla_m t = \frac{1}{N}(n^k \nabla_k N)\nu_m + N(\nabla_m \nabla_k t) n^k =$$
$$= \frac{1}{N}(n^k d_k N)\nu_m + N \nabla_m\left(\frac{1}{N}\nu_k\right) n^k = \frac{1}{N}(n^k d_k N)\nu_m - \frac{1}{N} d_m N \underbrace{\nu_k n^k}_{1} + \underbrace{(\nabla_m \nu_k) n^k}_{0}$$
$$= -\frac{1}{N} {}^{\shortparallel}\delta^k_m d_k N$$

properties

$$\nabla_a n^b = K_a{}^b + \nu_a a^b \qquad \Leftarrow \nabla_a n^b = {}^{\shortparallel}\delta^k_a \nabla_k n^b + {}^{\perp}\delta^k_a \nabla_k n^b = K_a{}^b + \nu_a \underbrace{n^k \nabla_k n^b}_{a^b}$$

$$\nabla_c \nu_b = K_{ab} + n_a a_b$$

$$\mathcal{L}_n n^a = 0$$

$$\mathcal{L}_n \nu_a = s a_a \qquad \Leftarrow \mathcal{L}_n \nu_a = n^k \nabla_k \nu_a + \underbrace{(\nabla_a n^k)\nu_k}_{0} = s a_a$$

$$\nabla_m n^m = k$$

$$\nabla_m a^m = {}^{\shortparallel}\nabla_m a^m - s a^2 \overset{*}{=} -\frac{s}{N} {}^{\shortparallel}\Box N \qquad \Leftarrow \nabla_m a^m = {}^{\shortparallel}\delta^k_m \nabla_k a^m + n^k(\nabla_k a^m)\nu_m = {}^{\shortparallel}\nabla_m a^m - \underbrace{n^n(\nabla_n \nu_m)}_{s a_m} a^m$$

$${}^{\shortparallel}\nabla_a a_b = -s\left[\frac{1}{N} {}^{\shortparallel}\nabla_a {}^{\shortparallel}\nabla_b N - a_a a_b\right] \qquad \Leftarrow {}^{\shortparallel}\nabla_a a_b = -s\, {}^{\shortparallel}\nabla_a\left(\frac{1}{N} {}^{\shortparallel}\nabla_b N\right) = -s\left[\frac{1}{N} {}^{\shortparallel}\nabla_a {}^{\shortparallel}\nabla_b N - \underbrace{\frac{1}{N^2}({}^{\shortparallel}\nabla_a N)({}^{\shortparallel}\nabla_b N)}_{a_a a_b}\right]$$

$$* \Leftarrow {}^{\shortparallel}\nabla_m a^m = -s q^{ab}\left[\frac{1}{N} {}^{\shortparallel}\nabla_a {}^{\shortparallel}\nabla_b N - a_a a_b\right] = -\frac{s}{N} {}^{\shortparallel}\Box N + s a^2$$

Foliation of spacetime - useful formulae

$$g_{ab} = s\,\nu_a\nu_b + q_{ab} \qquad g^{-1\,ab} = s\,n^a n^b + q^{ab}$$

$$\nu_a = N\,d_a t \qquad n^a\nu_a = 1 \qquad n^a d_a t = \frac{1}{N}$$

$$a^m = \nabla_n n^m \qquad \nabla_n \nu_m = s\,a_m = -\frac{1}{N}\,{}^{\parallel}d_m N \qquad {}^{\parallel}d_m N = -s N a_m$$

$$\overset{\parallel}{\nabla}_a a_b = -s\,\overset{\parallel}{\nabla}_a \overset{\parallel}{\nabla}_b \ln N = -s\left[\frac{1}{N}\overset{\parallel}{\nabla}_a \overset{\parallel}{\nabla}_b N - a_a a_b\right]$$

$$\nabla_m a^m = \overset{\parallel}{\nabla}_m a^m - s\,a^2 = -s\,\frac{1}{N}\,\overset{\parallel}{\Box} N$$

$$K_a{}^b = \nabla_{\parallel a} n^b \qquad K_{ab} = \nabla_{\parallel a}\nu_b \qquad K_{ab} = s\,K_{ab} \qquad K_{ab} = K_{ba}$$

$$\nabla_a \nu_b = K_{ab} + n_a a_b \qquad k = K_a{}^a \qquad K^2 = K_{ab}K^{ab}$$

$$\nabla_a n^b = K_a{}^b + \nu_a a^b \qquad K^2_{ab} = K_{am}K_{bn}q^{mn}$$

$$\mathcal{L}_n n^m = 0 \qquad \mathcal{L}_n \nu_m = s\,a_m$$

$$\nabla_{\parallel m} q_{ab} = -K_{ma}\nu_b - K_{mb}\nu_a \qquad \nabla_{\parallel m}\,{}^{\parallel}\delta^a_b = -K_m{}^a\nu_b - K_{mb}\nu^a$$

$$\nabla_\perp q_{ab} = -a_a\nu_b - \nu_a a_b \qquad \nabla_\perp\,{}^{\parallel}\delta^a_b = -a^a\nu_b - \nu^a a_b$$

$$\mathcal{L}_n q_{ab} = 2K_{ab} \qquad \mathcal{L}_n\,{}^{\parallel}\delta^a_b = -\nu^a a_b$$

$$\mathcal{L}_n q^{ab} = -2K^{ab} - \nu^a a^b - a^a\nu^b \qquad \mathcal{L}_{Nn}\,{}^{\parallel}\delta^a_b = 0$$

$$\mathcal{L}_n g_{ab} = -2K_{ab} + \nu_a a_b + a_a\nu_b$$

$$\mathcal{L}_n g^{ab} = -2K^{ab} - \nu^a a^b - a^a\nu^b$$

$$\nabla_n K_{ab} = {}^{\parallel}(\nabla_n K)_{ab} - \nu_a K_{bm}a^m - \nu_b K_{am}a^m$$

$$\mathcal{L}_n K_{ab} = {}^{\parallel}(\nabla_n K)_{ab} + 2K^2_{ab}$$

Covariant derivative of projector and metric

$$\nabla_m {}^{\parallel}\delta^a_b = -K_m{}^a \nu_b - K_{mb}\nu^a - \nu_m a^a \nu_b - \nu_m \nu^a a_b$$

proof: $\nabla_m {}^{\parallel}\delta^a_b = \nabla_m(\delta^a_b - n^a \nu_b) = -(\nabla_m n^a)\nu_b - n^a(\nabla_m \nu_b)$

$= -K_m{}^a \nu_b - \nu_m a^a \nu_b - K_{mb} n^a - n_m n^a a_b$

$= -K_m{}^a \nu_b - K_{mb}\nu^a - \nu_m a^a \nu_b - \nu_m \nu^a a_b$

$\Downarrow$

$$\nabla_\perp {}^{\parallel}\delta^a_b = -a^a \nu_b - \nu^a a_b$$

$$\nabla_{\parallel m} {}^{\parallel}\delta^a_b = -K_m{}^a \nu_b - K_{mb}\nu^a$$

$${}^{\parallel}(\nabla_m {}^{\parallel}\delta^a_b) = {}^{\parallel}\nabla_m {}^{\parallel}\delta^a_b = 0$$

$$\nabla_\perp q_{ab} = -a_a \nu_b - \nu_a a_b \qquad \nabla_\perp q^{ab} = -a^a \nu^b - \nu^a a^b$$

$$\nabla_{\parallel m} q_{ab} = -K_{ma}\nu_b - K_{mb}\nu_a \qquad \nabla_{\parallel m} q^{ab} = -K_m{}^a \nu^b - K_m{}^b \nu^a$$

Lie derivative of projector and metric

$$\mathcal{L}_n g_{ab} = 2K_{ab} + \nu_a a_b + a_a \nu_b$$

$\Uparrow$ $\mathcal{L}_n g_{ab} = \nabla_n g_{ab} + (\nabla_a n^k) g_{kb} + (\nabla_b n^k) g_{ak} = 0 + K_{ab} + \nu_a a_b + K_{ba} + \nu_b a_a$

$$\mathcal{L}_n g^{ab} = -2K^{ab} - \nu^a a^b - a^a \nu^b$$

$\Uparrow$ $\mathcal{L}_n g^{ab} = -g^{ak} g^{bl} \mathcal{L}_n g_{kl} \quad \Leftarrow \quad \delta^a_b = g^{ak} g_{bk} \quad \mathcal{L}_n \delta^a_b = 0$

$$\mathcal{L}_n q_{ab} = 2K_{ab}$$

$\Uparrow$ $\mathcal{L}_n q_{ab} = \mathcal{L}_n(g_{ab} - s\nu_a \nu_b) = 2K_{ab} + \nu_a a_b + a_a \nu_b - s^2 a_a \nu_b - s^2 \nu_a a_b = 2K_{ab}$

$$\mathcal{L}_n q^{ab} = -2K^{ab} - \nu^a a^b - a^a \nu^b = -2K^{ab} + \frac{1}{N} n^a\, {}^{\parallel}d^b N + \frac{1}{N} n^b\, {}^{\parallel}d^a N$$

$\Uparrow$ $\mathcal{L}_n q^{ab} = \mathcal{L}_n(g^{ab} - s n^a n^b) = \mathcal{L}_n g^{ab} = -2K^{ab} - \nu^a a^b - a^a \nu^b \qquad a^a = -s\frac{1}{N}\, {}^{\parallel}d^a N$

$$\mathcal{L}_n {}^{\parallel}\delta^a_b = -\nu^a a_b$$

$\Uparrow$ $\mathcal{L}_n {}^{\parallel}\delta^a_b = \mathcal{L}_n(\delta^a_b - n^a \nu_b) = -n^a \mathcal{L}_n \nu_b = -\nu^a a_b$

we have

$$K_{ab} = \frac{1}{2}\mathcal{L}_n q_{ab} = \frac{1}{2}\, {}^{\parallel}(\mathcal{L}_n g)_{ab}$$

$$\mathcal{L}_{Nn} {}^{\parallel}\delta^a_b = 0 \qquad \Leftarrow \mathcal{L}_{Nn} {}^{\parallel}\delta^a_b = N\mathcal{L}_n {}^{\parallel}\delta^a_b - \Omega_n dN\, {}^{\parallel}\delta^a_b = -N\nu^a a_b - n^a d_k N\, {}^{\parallel}\delta^k_b = 0$$

Derivatives of extrinsic curvature

$$\nabla_n K_{ab} = {}^{\parallel}(\nabla_n K)_{ab} - \nu_a K_{bm} a^m - \nu_b K_{am} a^m$$

$$\Uparrow \; {}^{\parallel}(\nabla_n K)_{ab} = {}^{\parallel}\delta_a^k \, {}^{\parallel}\delta_b^l \nabla_n K_{kl} = \nabla_n K_{ab} - \nu_a n^k \nabla_n K_{kb} - \nu_b n^k \nabla_n K_{ak} + \nu_a \nu_b \underbrace{n^k n^l \nabla_n K_{kl}}_{0}$$

$$= \nabla_n K_{ab} + \nu_a K_{bk} \nabla_n n^k + \nu_b K_{ak} \nabla_n n^k = \nabla_n K_{ab} + \nu_a K_{bm} a^m + \nu_b K_{am} a^m$$

$$\mathcal{L}_n K_{ab} = {}^{\parallel}(\nabla_n K)_{ab} + 2 K^2_{ab}$$

$$\Uparrow \; \mathcal{L}_n K_{ab} = \nabla_n K_{ab} + (\nabla_a n^k) K_{kb} + (\nabla_b n^k) K_{ak} =$$

$$= {}^{\parallel}(\nabla_n K)_{ab} - \nu_a K_{bm} a^m - \nu_b K_{am} a^m + (K_a{}^k + \nu_a a^k) K_{kb} + (K_b{}^k + \nu_b a^k) K_{ak}$$

$$= {}^{\parallel}(\nabla_n K)_{ab} + 2 K^2_{ab}$$

$$\nabla_n k = \mathcal{L}_n k = \nabla_m (k n^m) - k^2$$

$$\Uparrow \; \nabla_m (k n^m) = n^m \nabla_m k + k \nabla_m n^m = \nabla_n k + k^2 \qquad \mathcal{L}_n k = \nabla_n k$$

Normal components of curvature

$$R_{\perp a \perp b} = R_{\perp\| a \perp \| b} = R_{a \perp b \perp} = s R_{\perp a}{}^{\perp}{}_{b} =$$

$$= -\mathcal{L}_n K_{ab} + K^2_{ab} + \overset{\|}{\nabla}_a a_b - s\, a_a a_b$$

$$= -\mathcal{L}_n K_{ab} + K^2_{ab} - s \frac{1}{N} \overset{\|}{\nabla}_a \overset{\|}{\nabla}_b N$$

proof:

$$-R_{\perp a}{}^{\perp}{}_{b} = -n^k R_{ka}{}^{\ell}{}_{b} \nu_\ell = n^k \left[\nabla_k \nabla_a \nu_b - \nabla_a \nabla_k \nu_b \right]$$

$$= s n^k \nabla_k K_{ab} - s n^k \nabla_a K_{kb} + n^k \nabla_k (n_a a_b) - n^k \nabla_a (n_k a_b)$$

$$= s\overset{\|}{(\nabla_n K)}_{ab} - n_a K_{bm} a^m - n_b K_{am} a^m + s(\nabla_a n^k) K_{kb} + a_a a_b$$

$$+ s \nu_a n^k \nabla_k a_b - s \nabla_a a_b - (\nabla_a n_k) n^k a_b$$

$$= s\left[\mathcal{L}_n K_{ab} - 2K^2_{ab} + K_a{}^k K_{kb} + \nu_a \cancel{K_{bm}} a^m - \nu_a \cancel{K_{bm}} a^m + s\, a_a a_b \right.$$

$$\left. - \overset{\|}{\delta}{}^k_a (\nabla_k a_\ell)(\overset{\|}{\delta}{}^\ell_b + n^\ell \nu_b) - \nu_b K_{am} a^m \right]$$

$$= s\left[\mathcal{L}_n K_{ab} - K^2_{ab} + s\, a_a a_b - \overset{\|}{\nabla}_a a_b + \cancel{(\nabla_{\| a} n^\ell)} a_\ell \nu_b - \cancel{K_{am} a^m \nu_b} \right]$$

$$Ric_{\perp\perp} = k^2 - K^2 + \nabla_k (a^k - k n^k)$$

proof:

$$Ric_{\perp\perp} = R_{m\perp}{}^{m}{}_{\perp} = q^{ab} R_{\perp a \perp b} = -q^{ab} \mathcal{L}_n K_{ab} + K^2 + \overset{\|}{\nabla}_m a^m - s\, a^2$$

$$= -\mathcal{L}_n k + K_{ab} \mathcal{L}_n q^{ab} + K^2 + \nabla_m a^m$$

$$= K_{ab} (-2K^{ab} - \nu^a a^b - a^a \nu^b) + k^2 + K^2 + \nabla_m (a^m - k n^m)$$

$$= k^2 - K^2 + \nabla_m (a^m - k n^m)$$

$$Ric_{\perp \| a} = \overset{\|}{\nabla}_c K_a{}^c - \overset{\|}{\nabla}_a k$$

$$Ric_{\| a \| b} = -s \mathcal{L}_n K_{ab} + s(2K^2_{ab} - k K_{ab}) + \overset{\|}{Ric}_{ab} - \frac{1}{N} \overset{\|}{\nabla}_a \overset{\|}{\nabla}_b N$$

proof:

$$Ric_{\| a \| b} = s R_{\perp \| a \perp \| b} + \overset{\|}{Ric}_{ab} - s(k K_{ab} - K^2_{ab}) = \text{result}$$

$$\mathcal{R} = {}^{(3)}\mathcal{R} + s\left(k^2 - K^2\right) + 2s\,\nabla_k\left(a^k - k n^k\right)$$

proof :

$$\mathcal{R} = 2s\,Ric_{\perp\perp} + {}^{(3)}\mathcal{R} - s\left(k^2 - K^2\right) =$$

$$= \cancel{2}s\left(k^2 - K^2\right) + 2s\,\nabla_k\left(a^k - k n^k\right) + {}^{(3)}\mathcal{R}$$

$$Ein_{\perp\perp} = -\frac{s}{2}\,{}^{(3)}\mathcal{R} + \frac{1}{2}\left(k^2 - K^2\right)$$

$$Ein_{\perp\parallel a} = {}^{(3)}\nabla_m K_a{}^m - {}^{(3)}\nabla_a k$$

$$Ein_{\parallel a \parallel b} = -s\left[\mathcal{L}_n K_{ab} - (\mathcal{L}_n k)\,q_{ab} + k K_{ab} - 2K^2_{ab} - \frac{1}{2}\left(k^2 + K^2\right) q_{ab}\right]$$

$$+ {}^{(3)}Ein_{ab} - \frac{1}{N}\,{}^{(3)}\nabla_a {}^{(3)}\nabla_b N + \frac{1}{N}\left({}^{(3)}\Box N\right) q_{ab}$$

proof

$${}^{\parallel}Ein_{ab} = {}^{\parallel}Ric_{ab} - \frac{1}{2}\mathcal{R}\,q_{ab} =$$

$$= -s\left[\mathcal{L}_n K_{ab} + k K_{ab} - 2K^2_{ab}\right] + {}^{(3)}Ric_{ab} - \frac{1}{N}\,{}^{(3)}\nabla_a {}^{(3)}\nabla_b N$$

$$-s\left[\frac{1}{2}\left(k^2 - K^2\right) + \nabla_k a^k - \nabla_m\left(k n^m\right)\right] q_{ab} - \frac{1}{2}\,{}^{(3)}\mathcal{R}\,q_{ab}$$

$$= -s\left[\mathcal{L}_n K_{ab} - (\mathcal{L}_n k)\,q_{ab} + k K_{ab} - 2K^2_{ab} - \frac{1}{2}\left(k^2 + K^2\right) q_{ab}\right]$$

$$+ {}^{(3)}Ein_{ab} - \frac{1}{N}\,{}^{(3)}\nabla_a {}^{(3)}\nabla_b N + \frac{1}{N}\left({}^{(3)}\Box N\right) q_{ab}$$

where we used

$$\nabla_m a^m = -\frac{s}{N}\,{}^{(3)}\Box N \qquad \nabla_m\left(k n^m\right) = \mathcal{L}_n k + k^2 \qquad {}^{(3)}Ein = {}^{(3)}Ric - \frac{1}{2}\,{}^{(3)}\mathcal{R}\,q$$

Space + Time splitting of spacetime

foliation of spacetime M
by spatial slices Σ_t

embedding of space Σ
onto spatial slices Σ_t

$\iota_t : \Sigma \to \Sigma_t = \iota_t \Sigma \subset M$

provides identification of
points in different times

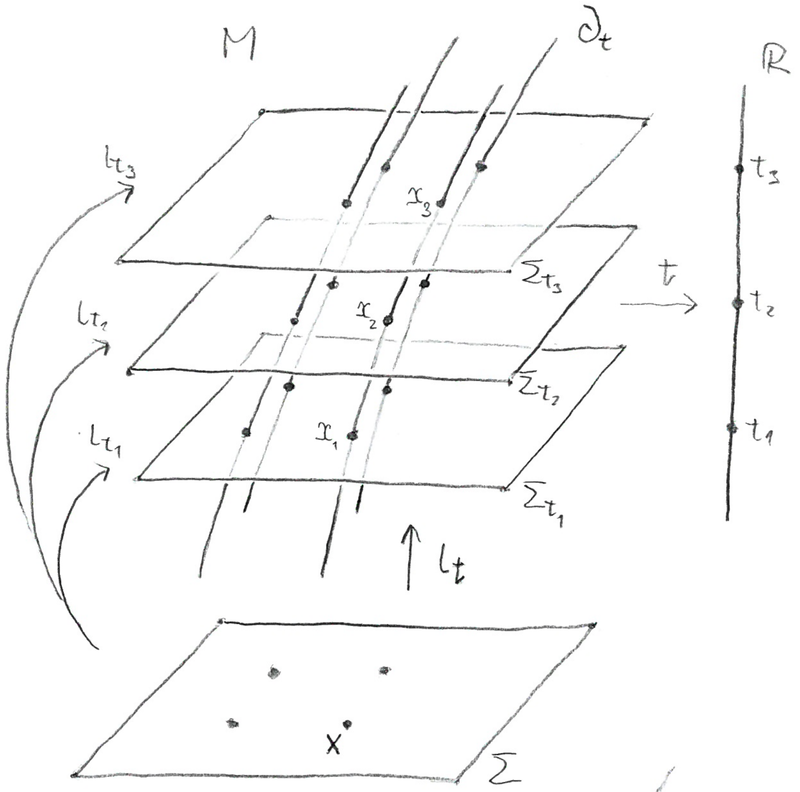
$t, x \to \mathcal{X} = \iota_t x$

time flow $\phi_{\Delta t}$

$\phi_{\Delta t} : M \to M$

$\Sigma_t \to \Sigma_{t+\Delta t}$

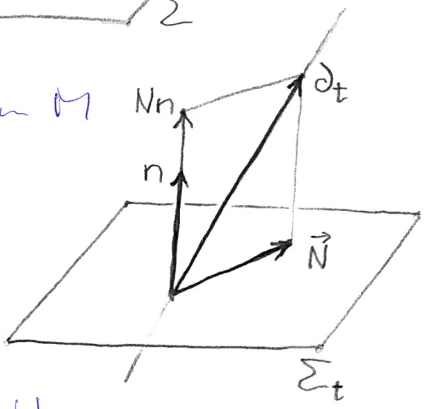
$\phi_{\Delta t} = \iota_{t+\Delta t} \circ \iota_t^{-1}$



generator of time flow

$\partial_t = \frac{D}{dt} \iota_t x$

$\mathcal{L}_{\partial_t} A = - \frac{d}{dt}\left(\phi_{t*} A \right)\Big|_{t=0}$ for a field A on M



parametrization of time flow

$\partial_t = Nn + \vec{N}$

$N \in \mathcal{F}M$ lapse $\quad \vec{N} \in T^{\parallel}M$ shift

$\partial_t \cdot dt = 1 \qquad \vec{N} \cdot dt = 0 \qquad n \cdot dt = \frac{1}{N} \qquad \nu = N dt$

reduction of tensor quantities to tangent tensor quant.

$A^{a\dots}_{b\dots} \in T^k_l M \to A^{\perp\dots}_{\perp\dots} \quad A^{\perp\dots}_{\parallel b\dots} \quad A^{\parallel a\dots}_{\perp\dots} \quad A^{\parallel a\dots}_{\parallel b\dots} \in T^{\parallel k}_{\parallel l} M$

all physical fields reduced to tangent fields
- these identified with fields on Σ

Identification of spatial objects on Σ with tangent objects on Σ_t

map $G^{-1} : \mathbb{T}^{\shortparallel}_{\shortparallel} M \to \mathbb{T}\Sigma$

vectors: $a^{\shortparallel} \in \mathbb{T}^{\shortparallel} M$ on $\Sigma_t \xrightarrow{\iota_{t*}^{-1}} a \in \mathbb{T}\Sigma$

not defined on $\mathbb{T}M$, but well-defined on $\mathbb{T}^{\shortparallel} M$

covectors: $\alpha_{\shortparallel} \in \mathbb{T}_{\shortparallel} M$ on $\Sigma_t \xrightarrow{\iota_t^*} \alpha \in \mathbb{T}^*\Sigma$

pull-back defined on $\mathbb{T}^* M \Rightarrow$ well-defined on $\mathbb{T}_{\shortparallel} M \subset \mathbb{T}^* M$

generalization on tensors ${}^{\shortparallel}A(x) \in \mathbb{T}^{\shortparallel}_{\shortparallel x} M \underset{x = \iota_t X}{\xrightarrow{G^{-1}}} A(t,x) \in \mathbb{T}_x \Sigma$

does not depend on a choice of normal subspace $\mathbb{T}^{\perp} M$

map $G_t : \mathbb{T}\Sigma \to \mathbb{T}^{\shortparallel}_{\shortparallel} M$ localized on Σ_t

vectors $a \in \mathbb{T}\Sigma \xrightarrow{\iota_{t*}} a^{\shortparallel} \in \mathbb{T}^{\shortparallel} M$ on Σ_t

push-forward - always defined

covectors $\alpha \in \mathbb{T}^*\Sigma \xrightarrow{\iota_{t*}} \alpha_{\shortparallel} \in \mathbb{T}_{\shortparallel} M$ on Σ_t

depends on a choice of normal subspace $\mathbb{T}^{\perp} M$

$\mathbb{T}_{\shortparallel} M$ realized as those covectors anihilating $\mathbb{T}^{\perp} M$

generalization on tensors $A(t,x) \in \mathbb{T}\Sigma \underset{x = \iota_t X}{\xrightarrow{G_t}} {}^{\shortparallel}A(x) \in \mathbb{T}^{\shortparallel}_{\shortparallel} M$

depends on a choice of normal subspace $\mathbb{T}^{\perp} M$ (for covectors)

Time derivative of tensor fields

t-dependent tensor field on Σ

$A(t,x) \to \dot{A}(t,x) = \frac{\partial}{\partial t} A(t,x)$

corresponding tensor field on M

${}^{\shortparallel}A(x) \to {}^{\shortparallel}\dot{A}(x) = {}^{\shortparallel}G_t \dot{A}(t,x) \qquad x = \iota_t X$

how to calculate ${}^{\shortparallel}\dot{A}$ directly on M ?

problem: G_t does not commute with time flow $\phi_{\Delta t}$

${}^{\shortparallel}A(x) \xrightarrow{G^{-1}} A(t,x) \xrightarrow{t \to t+\Delta t} A(t+\Delta t, x) \xrightarrow{G_t} {}^{\shortparallel}\tilde{A}(\phi_{\Delta t} x)$

${}^{\shortparallel}A$ and ${}^{\shortparallel}\tilde{A}$ are not related just by time flow

${}^{\shortparallel}\tilde{A} \neq \phi_{\Delta t *} {}^{\shortparallel}A$

since, in general,

$\phi_{\Delta t *} \mathbb{T}_{\shortparallel x} M \neq \mathbb{T}_{\shortparallel \phi_{\Delta t} x} M$

$\Rightarrow$ Lie derivative $\mathcal{L}_{\partial_t}$ on M does not correspond to $\dot{}$

Time-adjusted projection

orthogonal projection

$${}^{\perp}\delta^a_b = n^a \nu_b \qquad {}^{\parallel}\delta^a_b = \delta^a_b - {}^{\perp}\delta^a_b$$

time-flow adjusted projection

$${}^{(\perp)}\delta^a_b = \partial_t^a \, d_b t \qquad {}^{(\parallel)}\delta^a_b = \delta^a_b - {}^{(\perp)}\delta^a_b$$

alternative choice of the "normal" direction given by time flow

define an alternative representation of tangent objects as tensors from $\mathbb{T}^{(\parallel)}_{(\parallel)}M$

$${}^{(\parallel)}\sigma_t : \mathbb{T}\Sigma \to \mathbb{T}^{(\parallel)}_{(\parallel)}M \quad \text{localized on } \Sigma_t$$

vectors $\quad a \to a^{(\parallel)} = {}^{(\parallel)}\sigma_t a \in \mathbb{T}^{(\parallel)}M \qquad a^{(\parallel)} = a^{\parallel}$

covectors $\quad \alpha \to \alpha_{(\parallel)} = {}^{(\parallel)}\sigma_t \alpha \in \mathbb{T}_{(\parallel)}M \qquad \alpha_{(\parallel)} \neq \alpha_{\parallel}$

projectors

$${}^{(\perp)}\delta = \partial_t \, dt = (Nn + \vec{N})\tfrac{1}{N}\nu = n\nu + \vec{N} dt = {}^{\perp}\delta + \vec{N} dt$$

$${}^{(\parallel)}\delta = {}^{\parallel}\delta - \vec{N} dt$$

we have

$${}^{\parallel}\delta = {}^{\parallel}\delta \cdot {}^{(\parallel)}\delta \qquad {}^{(\parallel)}\delta = {}^{(\parallel)}\delta \cdot {}^{\parallel}\delta$$

representation of tangent objects

$\mathbb{T}^{(\parallel)}M = \mathbb{T}^{\parallel}M \qquad$ tangent vectors are unique

$$a \in \mathbb{T}\Sigma \to \begin{array}{l} a^{\parallel} = {}^{\parallel}\sigma_t a \in \mathbb{T}^{\parallel}M \\ a^{(\parallel)} = {}^{(\parallel)}\sigma_t a \in \mathbb{T}^{(\parallel)}M \end{array} \quad \text{but } a^{\parallel} = a^{(\parallel)}$$

$\mathbb{T}^{(\perp)}M \neq \mathbb{T}^{\perp}M \qquad$ normal directions are different $(\vec{N} \neq 0)$

$\mathbb{T}_{(\parallel)}M \neq \mathbb{T}_{\parallel}M \qquad$ tangent covectors are different (the anihilate normal directions)

$$\alpha \in \mathbb{T}^*\Sigma \to \begin{array}{l} \alpha_{\parallel} = {}^{\parallel}\sigma_t \alpha \in \mathbb{T}_{\parallel}M \\ \alpha_{(\parallel)} = {}^{(\parallel)}\sigma_t \alpha \in \mathbb{T}_{(\parallel)}M \end{array} \quad \text{in general, } \alpha_{\parallel} \neq \alpha_{(\parallel)} \ (\text{for } \vec{N} \neq 0)$$

but the projection back to $\mathbb{T}^*\Sigma$ is the same

$$\sigma^{-1}\alpha_{(\parallel)} = \sigma^{-1}\alpha_{\parallel} = \alpha \in \mathbb{T}^*\Sigma$$

relation of $\alpha_{\parallel}$ and $\alpha_{(\parallel)}$

$$\alpha_{(\parallel)} = {}^{(\parallel)}\delta \cdot \alpha_{\parallel} \quad \Leftarrow \quad \partial_t \cdot {}^{(\parallel)}\delta \cdot \alpha_{\parallel} = 0 \quad \forall a \in \mathbb{T}\Sigma \quad a^{(\parallel)} \cdot {}^{(\parallel)}\delta \cdot \alpha_{\parallel} \overset{\overset{a^{(\parallel)} = a^{\parallel}}{\downarrow}}{=} a^{\parallel} \cdot \alpha_{\parallel} - \overbrace{a^{\parallel} \cdot dt}^{0} \, \vec{N} \cdot \alpha_{\parallel} = a \cdot \alpha$$

$$\alpha_{\parallel} = {}^{\parallel}\delta \cdot \alpha_{(\parallel)} \quad \Leftarrow \quad n \cdot {}^{\parallel}\delta \cdot \alpha_{(\parallel)} = 0 \quad \forall a \in \mathbb{T}\Sigma \quad a^{\parallel} \cdot {}^{\parallel}\delta \cdot \alpha_{(\parallel)} = a^{(\parallel)} \cdot \alpha_{(\parallel)} + \underbrace{a^{(\parallel)} \cdot dt}_{0} \, \vec{N} \cdot \alpha_{(\parallel)} = a \cdot \alpha$$

relation of ${}^{\parallel}A \in \mathbb{T}^{\parallel}_{\parallel}M$ and ${}^{(\parallel)}A \in \mathbb{T}^{(\parallel)}_{(\parallel)}M$ corresponding to $A \in \mathbb{T}\Sigma$

$${}^{(\parallel)}A = {}^{(\parallel)}({}^{\parallel}A) \qquad {}^{\parallel}A = {}^{\parallel}({}^{(\parallel)}A)$$

Time adjusted projection commute with time flow

$$\phi_{\Delta t *} T_{(\parallel) x} M = T_{(\parallel) \phi_{\Delta t} x} M \qquad \phi_{\Delta t *}\, {}^{(\parallel)}\delta = {}^{(\parallel)}\delta \qquad \mathcal{L}_{\partial_t}\, {}^{(\parallel)}\delta = 0$$

since $\phi_{\Delta t *}\partial_t = \partial_t$, i.e. $\mathcal{L}_{\partial_t}\partial_t = 0$

we can write

$$\begin{tikzcd}
{}^{\parallel}A(x) \arrow[r, "{}^{(\parallel)}\delta"'] \arrow[rr, bend left, "\sigma^{-1}"] & {}^{(\parallel)}A(x) \arrow[r, "\sigma^{-1}"'] \arrow[rrr, bend right, "\phi_{\Delta t *}"'] & A(t,x) \arrow[r, "t\to t+\Delta t"'] & A(t+\Delta t, x) \arrow[r, "{}^{(\parallel)}\sigma_t"'] \arrow[rr, bend left, "{}^{\parallel}\sigma_t"] & {}^{(\parallel)}\tilde{A}(\phi_{\Delta t}x) \arrow[r, "{}^{\parallel}\delta"'] & {}^{\parallel}\tilde{A}(\phi_{\Delta t}x)
\end{tikzcd}$$

${}^{(\parallel)}A$ and ${}^{(\parallel)}\tilde{A}$ are related by time-flow $\phi_{\Delta t *}$

$${}^{(\parallel)}\tilde{A} = \phi_{\Delta t *}\, {}^{(\parallel)}A$$

time derivative - time-adjusted representation

$${}^{(\parallel)}(\dot{A}) = \mathcal{L}_{\partial_t}\, {}^{(\parallel)}A$$

time derivative - orthogonal representation

$${}^{\parallel}(\dot{A}) = {}^{\parallel (\parallel)}(\dot{A}) = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{(\parallel)}({}^{\parallel}A)) = {}^{\parallel (\parallel)}(\mathcal{L}_{\partial_t}\, {}^{\parallel}A) = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{\parallel}A)$$

$\Downarrow$

$${}^{\parallel}(\dot{A}) = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{\parallel}A)$$

for vectors the outer projection is not necessary

$${}^{\parallel}(\dot{a}) = \mathcal{L}_{\partial_t}\, {}^{\parallel}a$$

for covectors the outer projection is necessary

$${}^{\parallel}(\dot{\alpha}) = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{\parallel}\alpha) \qquad {}^{\perp}(\mathcal{L}_{\partial_t}\, {}^{\parallel}\alpha) \neq 0 \quad \text{(in general)}$$

for general tensors, we will use

$${}^{\parallel}\dot{A} \equiv {}^{\parallel}(\dot{A}) \equiv {}^{\parallel}\sigma_t \dot{A} = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{\parallel}A) \qquad {}^{\parallel}A = {}^{\parallel}\sigma_t A$$

time derivative and normal Lie-derivative

$${}^{\parallel}\dot{A} = N\, {}^{\parallel}(\mathcal{L}_n\, {}^{\parallel}A) + \mathcal{L}^{\parallel}_{\vec{N}}\, {}^{\parallel}A$$

$${}^{\parallel}(\mathcal{L}_n\, {}^{\parallel}A) = \frac{1}{N}\left({}^{\parallel}\dot{A} - \mathcal{L}^{\parallel}_{\vec{N}}\, {}^{\parallel}A\right)$$

where $\mathcal{L}^{\parallel}_{\vec{N}}$ is representation of Lie der. on Σ - see %

proof:

$${}^{\parallel}\dot{A} = {}^{\parallel}(\mathcal{L}_{\partial_t}\, {}^{\parallel}A) = {}^{\parallel}(\mathcal{L}_{Nn+\vec{N}}\, {}^{\parallel}A) = {}^{\parallel}(N\mathcal{L}_n\, {}^{\parallel}A + \Omega_{n\,dN}\, {}^{\parallel}A + \mathcal{L}_{\vec{N}}\, {}^{\parallel}A)$$

$$= N\, {}^{\parallel}(\mathcal{L}_n\, {}^{\parallel}A) + \Omega\, \underbrace{{}^{\parallel}(n\,dN)}_{0}\, {}^{\parallel}A + \mathcal{L}^{\parallel}_{\vec{N}}\, {}^{\parallel}A$$

we need: $\mathcal{L}_{fc}A = f\mathcal{L}_c A + \Omega_{c\,df}A$

$\mathcal{L}^{\parallel}_c\, {}^{\parallel}A = {}^{\parallel}(\mathcal{L}_{{}^{\parallel}c}\, {}^{\parallel}A)$

Representation of Lie derivative on Σ

Lie derivative on Σ

$$\mathcal{L}_c A \qquad A \in \mathrm{Sect}\, \mathbb{T}^k_l \Sigma, \quad c \in \mathrm{Sect}\, \mathbb{T}\Sigma$$

representation in $\mathbb{T}^{\shortparallel}_{\shortparallel} M$

$$\mathcal{L}^{\shortparallel}_{c}\, {}^{\shortparallel}\!A \qquad {}^{\shortparallel}\!A \in \mathrm{Sect}\, \mathbb{T}^{\shortparallel k}_{\shortparallel l} M \text{ on } \Sigma_t, \quad {}^{\shortparallel}c \in \mathrm{Sect}\, \mathbb{T}^{\shortparallel} M \text{ on } \Sigma_t$$

Lie derivative on M

$$\mathcal{L}_{{}^{\shortparallel}c}\, {}^{\shortparallel}\!A \qquad {}^{\shortparallel}\!A \in \mathrm{Sect}\, \mathbb{T}^{\shortparallel k}_{\shortparallel l} M \text{ extension around } \Sigma_t, \quad {}^{\shortparallel}c \in \mathrm{Sect}\, \mathbb{T}^{\shortparallel} M \text{ extension around } \Sigma_t$$

for example ${}^{\shortparallel}c = {}^{\shortparallel}G_t\, c \qquad {}^{\shortparallel}\!A = {}^{\shortparallel}G_t\, A$

where c and A can be t-dependent

relation

$$\mathcal{L}^{\shortparallel}_{{}^{\shortparallel}c}\, {}^{\shortparallel}\!A = {}^{\shortparallel}\big(\mathcal{L}_c\, {}^{\shortparallel}\!A\big)$$

comments:

- normal directions $\sim n$, i.e. $\mathbb{T}^{\perp} M$, does not commute with flow φ_s generated by ${}^{\shortparallel}c$ around Σ_t
- tangent directions $\mathbb{T}^{\shortparallel} M$ do commute with flow φ_s
- cotangent directions $\mathbb{T}_{\shortparallel} M$ do not commute with flow φ_s

proof:

$$\mathcal{L}_c A = \nabla_c A - \Omega_{\nabla c} A \text{ on } \Sigma \quad \Rightarrow \quad \mathcal{L}^{\shortparallel}_c\, {}^{\shortparallel}\!A = \nabla^{\shortparallel}_{{}^{\shortparallel}c}\, {}^{\shortparallel}\!A - \Omega_{\nabla^{\shortparallel}\, {}^{\shortparallel}c}\, {}^{\shortparallel}\!A$$

$${}^{\shortparallel}\big(\mathcal{L}_c\, {}^{\shortparallel}\!A\big) = {}^{\shortparallel}\big(\nabla_{{}^{\shortparallel}c}\, {}^{\shortparallel}\!A - \Omega_{\nabla {}^{\shortparallel}c}\, {}^{\shortparallel}\!A\big) = {}^{\shortparallel}\big(\nabla_c\, {}^{\shortparallel}\!A\big) - \Omega_{{}^{\shortparallel}(\nabla {}^{\shortparallel}c)}\, {}^{\shortparallel}\!A$$

$$= \nabla^{\shortparallel}_{{}^{\shortparallel}c}\, {}^{\shortparallel}\!A - \Omega_{\nabla^{\shortparallel}\, {}^{\shortparallel}c}\, {}^{\shortparallel}\!A = \mathcal{L}^{\shortparallel}_c\, {}^{\shortparallel}\!A$$

Time derivative of q_{ab} and K_{ab}

$$''\mathring{A} = N\,''(\mathcal{L}_n\,''A) + \mathcal{L}^{''}_{\vec{N}}\,''A$$

metric

$$q_{ab} = {}''g_{ab} \qquad q^{ab} = {}''g^{ab}$$

$\Downarrow$

$$\dot{q}_{ab} = N\,''(\mathcal{L}_n q)_{ab} + \mathcal{L}^{''}_{\vec{N}} q_{ab} = 2NK_{ab} + \mathcal{L}^{''}_{\vec{N}} q_{ab}$$

$$= 2NK_{ab} + {}^{''}\nabla_a \vec{N}_b + {}^{''}\nabla_b \vec{N}_a$$

$$^{\#}\dot{q}^{ab} = N\,''(\mathcal{L}_n {}^{\#}q)^{ab} + \mathcal{L}^{''}_{\vec{N}}\,{}^{\#}q^{ab} = N\,''\Big(-2K^{ab} + \frac{1}{N} n^a\,''d^b N + \frac{1}{N} n^b\,''d^a N\Big) + \mathcal{L}^{''}_{\vec{N}} q^{ab}$$

$$= -2NK^{ab} - {}^{''}\nabla^a \vec{N}^b - {}^{''}\nabla^b \vec{N}^a$$

extri-sic curvature

$$K_{ab} = {}''K_{ab}$$

$$\dot{K}_{ab} = N\,''(\mathcal{L}_n K)_{ab} + \mathcal{L}^{''}_{\vec{N}} K_{ab}$$

$$= N\,''(\nabla_n K_{ab} + \nu_a K_{bk} a^k + \nu_b K_{ak} a^k + 2K^2_{ab}) + \mathcal{L}^{''}_{\vec{N}} K_{ab}$$

$$= N\,''(\nabla_n K)_{ab} + 2NK^2_{ab} + \mathcal{L}^{''}_{\vec{N}} K_{ab}$$

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - \mathcal{L}^{''}_{\vec{N}} q_{ab}\right)$$

Splitting of densities

$\Box \in \tilde{\mathcal{F}} M$ density on M

${}^{\parallel}\Box \in \tilde{\mathcal{F}} \Sigma_t$ orthogonal spatial projection
= density on Σ_t

$${}^{\parallel}\Box[e_i] = \Box[n, e_i]$$

frame in ${}^{\parallel}\mathbb{T}\Sigma_t$ frame in $\mathbb{T}M$

$$\Box = |\nu|\ {}^{\parallel}\Box \qquad |\nu|[n] = 1$$

density in time direction density in spatial directions

${}^{(\parallel)}\Box \in \tilde{\mathcal{F}} \Sigma_t$ time-adjusted spatial projection
= density on Σ_t

$${}^{(\parallel)}\Box[e_i] = \Box[\partial_t, e_i]$$

frame in ${}^{\parallel}\mathbb{T}\Sigma_t$ frame in $\mathbb{T}M$

$$\Box = \mathrm{d}t\ {}^{(\parallel)}\Box \qquad \mathrm{d}t \equiv |\mathrm{d}t| \qquad \mathrm{d}t[\partial_t] = 1$$

density in time direction density in spatial directions

relation:

$${}^{(\parallel)}\Box = N\ {}^{\parallel}\Box$$

proof: $\Box = |\nu|\ {}^{\parallel}\Box = N\mathrm{d}t\ {}^{\parallel}\Box = \mathrm{d}t\ {}^{(\parallel)}\Box \quad \Rightarrow \quad {}^{(\parallel)}\Box = N\ {}^{\parallel}\Box$

$${}^{(\parallel)}\Box[e_i] = \Box[\partial_t, e_i] = \Box[Nn + \vec{N}, e_i] = \Box[Nn, e_i] = N\Box[n, e_i] = N\ {}^{\parallel}\Box[e_i]$$

densities on Σ and on Σ_t are in unique correspondence
- defined by an action on tangent vectors which are unique

$$\tilde{\mathcal{F}}\Sigma \leftrightarrow \tilde{\mathcal{F}}\Sigma_t$$

metric volume element

$$g^{\frac{1}{2}} = |\nu|\ \underset{\smile}{g}^{\frac{1}{2}} = N\mathrm{d}t\ \underset{\smile}{g}^{\frac{1}{2}}$$

$${}^{\parallel}g^{\frac{1}{2}} = \underset{\smile}{g}^{\frac{1}{2}} \qquad {}^{(\parallel)}g^{\frac{1}{2}} = N\ \underset{\smile}{g}^{\frac{1}{2}}$$

Time derivative of densities

$m \in \tilde{\mathcal{F}}\Sigma$ t-dependent density on Σ

$\dot{m} = \frac{\partial}{\partial t} m \in \tilde{\mathcal{F}}\Sigma$ time derivative on Σ

$\dot{m} \in \hat{\mathcal{F}}\Sigma_t$ unique density on Σ_t corresponding to $\dot{m}$

time shift $t \mapsto t+\Delta T$ on Σ is equivalent to time flow $\phi_{\Delta T}$ on M

$$\dot{m} = \mathcal{L}_{\partial_t} m$$

$\mathcal{L}_{\partial_t} m$ is a strange derivative acting on densities on Σ_t
better to reformulate to action on densities on M

dt commutes with time flow $\phi_{\Delta T}$ $\Leftrightarrow \mathcal{L}_{\partial_t} dt = 0$

time-adjusted splitting commutes with time flow $\phi_{\Delta T}$

$$\square = dt\, {}^{(\parallel)}\square \qquad \text{density on } M$$

$$\mathcal{L}_{\partial_t}\square = \mathcal{L}_{\partial_t}(dt\, {}^{(\parallel)}\square) = dt\, \mathcal{L}_{\partial_t} {}^{(\parallel)}\square = dt\, ({}^{(\parallel)}\square)^{\circ}$$

$$\Downarrow$$

$$({}^{(\parallel)}\square)^{\circ} = {}^{(\parallel)}(\mathcal{L}_{\partial_t}\square)$$

orthogonal splitting

$$({}^{\parallel}\square)^{\circ} = {}^{\parallel}(\mathcal{L}_{\partial_t}\square) - \frac{\dot{N}}{N}\, {}^{\parallel}\square$$

proof: ${}^{(\parallel)}\square = N\, {}^{\parallel}\square$

$({}^{(\parallel)}\square)^{\circ} = {}^{(\parallel)}(\mathcal{L}_{\partial_t}\square) \Rightarrow (N\, {}^{\parallel}\square)^{\circ} = N\, {}^{\parallel}(\mathcal{L}_{\partial_t}\square) \Rightarrow N({}^{\parallel}\square)^{\circ} + \dot{N}\, {}^{\parallel}\square = N\, {}^{\parallel}(\mathcal{L}_{\partial_t}\square)$

$\Rightarrow ({}^{\parallel}\square)^{\circ} = {}^{\parallel}(\mathcal{L}_{\partial_t}\square) - \frac{\dot{N}}{N}\, {}^{\parallel}\square$

metric volume element

$$(g^{\frac{1}{2}})^{\circ} = (Nk + \overset{\parallel}{\nabla}\cdot\vec{N})\, g^{\frac{1}{2}}$$

proof 1:

$$(g^{\frac{1}{2}})^{\circ} = ({}^{\parallel}g^{\frac{1}{2}})^{\circ} = {}^{\parallel}(\mathcal{L}_{\partial_t} g^{\frac{1}{2}}) - \frac{\dot{N}}{N}\, {}^{\parallel}g^{\frac{1}{2}} = {}^{\parallel}(\nabla\cdot(\partial_t g^{\frac{1}{2}})) - \frac{\dot{N}}{N} g^{\frac{1}{2}}$$

$$= (\nabla\cdot\partial_t - \frac{\dot{N}}{N})\, g^{\frac{1}{2}} = (\nabla\cdot(Nn) + \nabla\cdot\vec{N} - \frac{\dot{N}}{N})\, g^{\frac{1}{2}}$$

$$= (n\cdot\nabla N + N\nabla\cdot n + \overset{\parallel}{\nabla}\cdot\vec{N} + n\cdot(\nabla\vec{N})\cdot\nu - \frac{1}{N}\dot{N})\, g^{\frac{1}{2}}$$

$$= (\frac{1}{N} Nn\cdot dN - n\cdot(\nabla\nu)\cdot\vec{N} - \frac{1}{N}\dot{N} + Nk + \overset{\parallel}{\nabla}\cdot\vec{N})\, g^{\frac{1}{2}} \qquad \Leftarrow n\cdot\nabla\nu = -\frac{1}{N}\, {}^{\parallel}dN$$

$$= (\frac{1}{N}\cancel{(Nn+\vec{N})\cdot dN - \frac{1}{N}\partial_t dN} + Nk + \overset{\parallel}{\nabla}\cdot\vec{N})\, g^{\frac{1}{2}}$$

proof 2:

$$(g^{\frac{1}{2}})^{\circ} = (\mathrm{Det}^{\frac{1}{2}} q)^{\circ} = \frac{1}{2} q^{ab}\dot{q}_{ab}\, g^{\frac{1}{2}} = \frac{1}{2} q^{ab}(2NK_{ab} + \overset{\parallel}{\mathcal{L}}_{\vec{N}} q_{ab})\, g^{\frac{1}{2}}$$

$$= (Nk + \overset{\parallel}{\nabla}\cdot\vec{N})\, g^{\frac{1}{2}}$$

Lagrangien formalism

action

$$S_{GR} = \frac{1}{\varkappa}\int_{\Omega} (R - 2\Lambda)\, g^{\frac{1}{2}} + \frac{2s}{\varkappa}\int_{\partial_{out}\Omega} k\, g^{\frac{1}{2}}$$

sandwich domain

$$\Omega = \langle \Sigma_f | \Sigma_i \rangle \qquad t \in \langle t_i, t_f \rangle$$

$$S_{GR} = \frac{1}{\varkappa}\int_{t_i}^{t_f}\int_{\Sigma_t} \left(-s(K^2 - k^2) + {}^{\shortparallel}R - 2\Lambda\right) N g^{\frac{1}{2}}\, dt$$

$$+ \frac{2}{\varkappa}\int_{\Omega} \nabla_m (a^m - k n^m)\, g^{\frac{1}{2}} + \frac{2s}{\varkappa}\int_{\Sigma_f - \Sigma_i} k\, g^{\frac{1}{2}}$$

$$= \int_{t_i}^{t_f} \frac{1}{\varkappa}\int_{\Sigma_t} \left(-s(K^2 - k^2) + {}^{\shortparallel}R - 2\Lambda\right) N g^{\frac{1}{2}}\, dt$$

$$+ \frac{2s}{\varkappa}\int_{\partial\Omega} (a^m - k n^m)\, \nu_{out\, m}\, g^{\frac{1}{2}} + \frac{2s}{\varkappa}\int_{\Sigma_f - \Sigma_i} k\, g^{\frac{1}{2}}$$

$$= \int_{t_i}^{t_f} \frac{1}{\varkappa}\int_{\Sigma_t} \left(-s(K^2 - k^2) + {}^{\shortparallel}R - 2\Lambda\right) N g^{\frac{1}{2}}\, dt$$

Lagrangien

$$L_{GR} = \frac{1}{\varkappa}\int_{\Sigma_t} \left(-s(K^2 - k^2) + {}^{\shortparallel}R - 2\Lambda\right) N g^{\frac{1}{2}}$$

velocity

$$\dot{q}_{ab} = {}^{\shortparallel}(\mathcal{L}_{\partial_t} q)_{ab} = N\, {}^{\shortparallel}(\mathcal{L}_n q)_{ab} + {}^{\shortparallel}\mathcal{L}_{\vec{N}}\, q_{ab}$$

$$= 2N K_{ab} + {}^{\shortparallel}\mathcal{L}_{\vec{N}}\, q_{ab}$$

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - {}^{\shortparallel}\mathcal{L}_{\vec{N}}\, q_{ab}\right)$$

$$\frac{\partial K_{ab}}{\partial \dot{q}_{ab}} = \frac{1}{2N}\, {}^{\shortparallel}\delta^{(a|}_{(m}\, \delta^{|b)}_{n)}$$

$$k = q^{ab} K_{ab}$$

$$\frac{\partial k}{\partial \dot{q}_{ab}} = \frac{\partial k}{\partial K} \frac{\partial K}{\partial \dot{q}_{ab}} = \frac{1}{2N} q^{ab} \qquad \frac{\partial k^2}{\partial \dot{q}_{ab}} = \frac{1}{N} k q^{ab}$$

$$K^2 = q^{ac} q^{bd} K_{ab} K_{cd}$$

$$\frac{\partial K^2}{\partial \dot{q}_{ab}} = 2 K^{mn} \frac{\partial K_{mn}}{\partial \dot{q}_{ab}} = \frac{1}{N} K^{ab}$$

momentum

$$\Pi^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = -s \frac{1}{\varkappa} N g^{\frac{1}{2}} \frac{1}{N} (K^{ab} - k q^{ab}) = -s \frac{1}{\varkappa} (K^{ab} - k q^{ab}) g^{\frac{1}{2}}$$

$$\pi = \Pi^{ab} q_{ab} = \frac{s}{\varkappa} (n-1) k g^{\frac{1}{2}} \qquad k = s \varkappa \frac{1}{n-1} \pi g^{-\frac{1}{2}}$$

$$K_{ab} = -s \varkappa \left(\Pi_{ab} - \frac{1}{n-1} \pi q_{ab} \right) g^{-\frac{1}{2}}$$

$$K^2 = \varkappa^2 \left(\mathrm{Tr}\,\Pi^2 - \frac{n-2}{(n-1)^2} \pi^2 \right) g^{-1} \qquad \mathrm{Tr}\,\Pi^2 = \Pi_{ab} \Pi^{ab}$$

$$K^2 - k^2 = \varkappa^2 \left(\mathrm{Tr}\,\Pi^2 - \frac{1}{n-1} \pi^2 \right) g^{-1}$$

$$\dot{q}_{ab} = -s \varkappa 2N \left(\Pi_{ab} - \frac{1}{n-1} \pi q_{ab} \right) g^{-\frac{1}{2}} + \mathcal{L}_{\vec{N}} q_{ab}$$

$$\Pi^{ab} \dot{q}_{ab} = -s \varkappa N \left(2 \mathrm{Tr}\,\Pi^2 - \frac{2}{n-1} \pi^2 \right) g^{-\frac{1}{2}} + 2 \Pi^{ab} \nabla_a \vec{N}_b$$

Hamiltonian

- without Lagr. multipl. for $\dot{N}$ and $\dot{\vec{N}}$

$$H_{GR} = \int_\Sigma \Pi^{ab} \dot{q}_{ab} - L_{GR}$$

$$= \int_\Sigma N \left[-s \varkappa \left(\mathrm{Tr}\,\Pi^2 - \frac{1}{n-1} \pi^2 \right) g^{-\frac{1}{2}} - \frac{1}{\varkappa} (R - 2\Lambda) g^{\frac{1}{2}} \right]$$

$$-2 \int_\Sigma \vec{N}^a \left(\nabla_m \Pi^{mb} \right) q_{ab}$$

Hamiltonian

$$H_{GR} = \int_\Sigma \pi^{ab} \dot{q}_{ab} - L_{GR}$$

$$= \int_\Sigma \Big[-s\varkappa N \left(2 \mathrm{Tr}\,\pi^2 - \frac{2}{n-1} \pi^2 \right) g^{-\frac{1}{2}} + 2\pi^{ab} \overset{(4)}{\nabla}_a \vec{N}_b$$

$$- s \frac{\varkappa^2}{\varkappa} N \left(- \mathrm{Tr}\,\pi^2 + \frac{2}{n-1} \pi^2 \right) g^{-\frac{1}{2}} - \frac{1}{\varkappa} N \left(\overset{(4)}{R} - 2\Lambda \right) g^{\frac{1}{2}} \Big]$$

$$= \int_\Sigma N \Big[-s\varkappa \left(\mathrm{Tr}\,\pi^2 - \frac{1}{n-1} \pi^2 \right) g^{-\frac{1}{2}} - \frac{1}{\varkappa} \left(\overset{(4)}{R} - 2\Lambda \right) g^{\frac{1}{2}} \Big]$$

$$- \int_\Sigma 2 \vec{N}^a \left(\nabla_m \pi^{mb} \right) q_{ab}$$